6.3 Bernoulli Trials

Example 6.48. Consider the following random experiments

(a) Flip a coin 10 times. We are interested in the number of heads obtained.

(b) Of all bits transmitted through a digital transmission channel, 10% are received in error. We are interested in the number of bits in error in the next five bits transmitted.

(c) A multiple-choice test contains 10 questions, each with four choices, and you guess at each question. We are interested in the number of questions answered correctly.

These examples illustrate that a general probability model that includes these experiments as particular cases would be very useful.

Example 6.49. Each of the random experiments in Example 6.48 can be thought of as consisting of a series of repeated, random trials. In all cases, we are interested in the number of trials that meet a specified criterion. The outcome from each trial either meets the criterion or it does not; consequently, each trial can be summarized as resulting in either a success or a failure.

Definition 6.50. A ***Bernoulli trial*** involves performing an experiment once and noting whether a particular event A occurs.

The outcome of the Bernoulli trial is said to be

(a) a "success" if A occurs and

(b) a "failure" otherwise.

We may view the outcome of a single Bernoulli trial as the outcome of a toss of an unfair coin for which the probability of heads (success) is $p = P(A)$ and the probability of tails (failure) is $1 - p$.

- Only one important parameter:

$$p = \text{success probability (probability of "success")}$$

- The labeling ("success" and "failure") is not meant to be literal and sometimes has nothing to do with the everyday meaning of the words. We can just as well use "H and T", "A and B", or "1 and 0".

Example 6.51. Examples of Bernoulli trials: Flipping a coin, deciding to vote for candidate A or candidate B, giving birth to a boy or girl, buying or not buying a product, being cured or not being cured, even dying or living are examples of Bernoulli trials.

- Actions that have multiple outcomes can also be modeled as Bernoulli trials if the question you are asking can be phrased in a way that has a yes or no answer, such as "Did the dice land on the number 4?".

Definition 6.52. (Independent) ***Bernoulli Trials*** = a Bernoulli trial is repeated many times.

(a) It is usually[28] assumed that the trials are independent. This implies that the outcome from one trial has no effect on the outcome to be obtained from any other trial.

(b) Furthermore, it is often reasonable to assume that the probability of a success in each trial is constant.

An outcome of the complete experiment is a sequence of successes and failures which can be denoted by a ***sequence of ones and zeroes***.

Example 6.53. Toss ***unfair coin*** n $\leftarrow 5$ times.

- The overall sample space is $\Omega = \{H, T\}^n$.
  - There are 2^n elements. Each has the form $(\omega_1, \omega_2, \ldots, \omega_n)$ where ω_i = H or T.
- The n tosses are independent. Therefore,

$$P(\{\text{HHHTT}\}) = \underbrace{P(\{H\})}_{p}P(\{H\})P(\{H\})P(\{T\})P(\{T\}) = ppp(1-p)(1-p)$$

$$= p^3(1-p)^2$$

[28]Unless stated otherwise or having enough evidence against, assume the trials are independent.

Example 6.54. What is the probability of two failures and three successes in five Bernoulli trials with success probability p.

Let's represent success and failure by 1 and 0, respectively. The outcomes with three successes in five trials are listed below:

Outcome	Corresponding probability
11100	$p \times p \times p \times (1-p) \times (1-p) = p^3(1-p)^2$
11010	$p \times p \times (1-p) \times p \times (1-p) = p^3(1-p)^2$
11001	"
10110	"
10101	"
10011	"
01110	"
01101	"
01011	"
00111	"

Among the 5 positions, choose 2 positions for 0s.
Among the 5 positions choose 3 positions for 1s.
$\binom{5}{3} = \binom{5}{2} = 10$

We note that the probability of each outcome is a product of five probabilities, each related to one Bernoulli trial. In outcomes with three successes, three of the probabilities are p and the other two are $1-p$. Therefore, each outcome with three successes has probability $(1-p)^2p^3$.

There are 10 of them. Hence, the total probability is $10(1-p)^2p^3$

6.55. The probability of exactly k successes in n bernoulli trials is

$$\binom{n}{k}(1-p)^{n-k}p^k.$$

Example 6.56. Consider a particular disease with prevalence $P(D) = 10^{-4}$: when a person is selected randomly from the general population, the probability that (s)he has this disease is 10^{-4} or 1-in-n where $n = 10^4$.

Suppose we randomly select $n = 10^4$ people from the general population. What is the chance that we find at least one person with this disease? $\approx 63\%$

0.632139895356 7030

$1-\frac{1}{e}$ 0.6321205588...

$p = \frac{1}{n}$

Example 6.57. At least one occurrence of a 1-in-n-chance event in n repeated trials:

$n = 2$: "Obtaining at least one H in 2 tosses of a fair coin" $\Rightarrow A$

$p = P(\{H\}) = \frac{1}{2}$ $\qquad$ $P(A) = \frac{3}{4} = 0.75$

$n = 6$: "obtaining at least one "six" in 6 tosses of a fair dice" $\Rightarrow A$

$$P(A) = 1 - \left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right) = 1 - \left(1 - \tfrac{1}{6}\right)^6$$

General n: $P(A) = 1 - \left(1 - \frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} 1 - e^{-1} = 1 - \frac{1}{e} \approx 0.6321 \approx 63\%.$

Two important results from calculus:

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

p(exactly 2 occurrences)

$$\binom{n}{2} p^2 (1-p)^{n-2}$$

$$= \binom{n}{2}\left(\frac{1}{n}\right)^2\left(1 - \frac{1}{n}\right)^{n-2}$$

$$= \frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 \left(1 - \frac{1}{n}\right)^{n-2}$$

$\frac{n(n-1)}{2}\left(\frac{1}{n}\right)^2 \xrightarrow{n\to\infty} \frac{1}{2}$

$\left(1 - \frac{1}{n}\right)^{n-2} = \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^{-2}$; $\left(1 - \frac{1}{n}\right)^n \xrightarrow{n\to\infty} \frac{1}{e}$; $\left(1 - \frac{1}{n}\right)^{-2} \to 1$

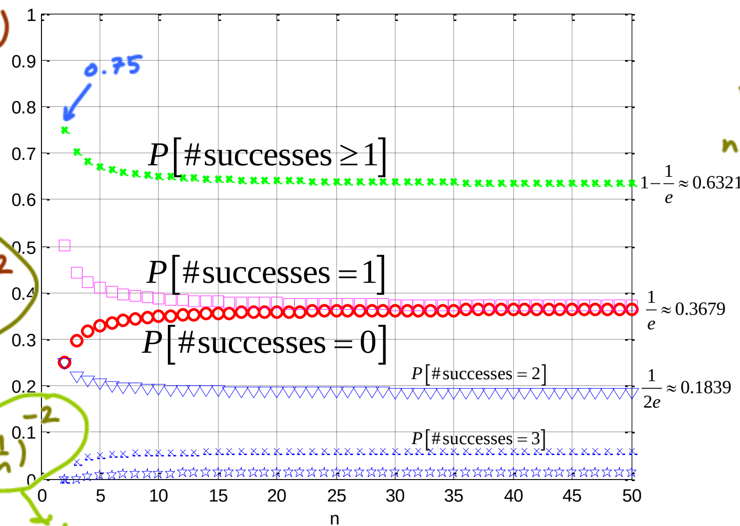
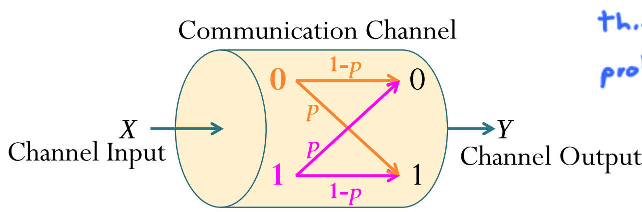


Figure 17: Number of occurrences of 1-in-n-chance event in n repeated Bernoulli trials

$$\lim_{n\to\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0}{b_k n^k + b_{k-1} n^{k-1} + \cdots + b_1 n + b_0} = \frac{a_k}{b_k}$$

Example 6.58. ***Digital communication over unreliable channels***: Consider a digital communication system through the **binary symmetric channel** (BSC) discussed in Example 6.18. We repeat its compact description here.



Suppose we put 010110 into this channel. What is the probability that we get 011010 as output?

$(1-p)(1-p)\,p\,p\,(1-p)(1-p)$

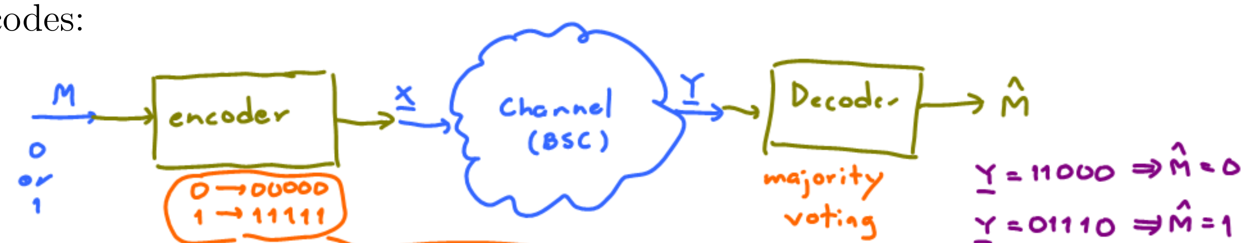
$(1-p)^4 p^2$

Again this channel can be described as a channel that introduces random bit errors with probability p. This p is called the crossover probability.

A crude digital communication system would put binary information into the channel directly; the receiver then takes whatever value that shows up at the channel output as what the sender transmitted. Such communication system would directly suffer bit error probability of p.

In situation where this error rate is not acceptable, error control techniques are introduced to reduce the error rate in the delivered information.

One method of reducing the error rate is to use error-correcting codes:



A simple error-correcting code is the ***repetition code***. Example of such code is described below:

- At the transmitter, the "encoder" box performs the following task:
  - To send a 1, it will send 11111 through the channel.
  - To send a 0, it will send 00000 through the channel.

- When the five bits pass through the channel, it may be corrupted. Assume that the channel is binary symmetric and that it acts on each of the bit independently.

- At the receiver, we (or more specifically, the decoder box) get 5 bits, but some of the bits may be changed by the channel.

  To determine what was sent from the transmitter, the receiver apply the ***majority rule***: Among the 5 received bits,
  - if $\#1 > \#0$, then it claims that "1" was transmitted,
  - if $\#0 > \#1$, then it claims that "0" was transmitted.

Two ways to calculate the probability of error:

(a) (transmission) error occurs if and only if the number of bits in error are ≥ 3.

$$P(\mathcal{E}) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5\cancel{(1-p)^0}$$

(b) (transmission) error occurs if and only if the number of bits *not* in error are ≤ 2.

$$P(\mathcal{E}) = \binom{5}{0}(1-p)^0p^5 + \binom{5}{1}(1-p)^1p^4 + \binom{5}{2}(1-p)^2p^3$$

with $p = 0.01$

$P(\mathcal{E}) \approx 10^{-5}$

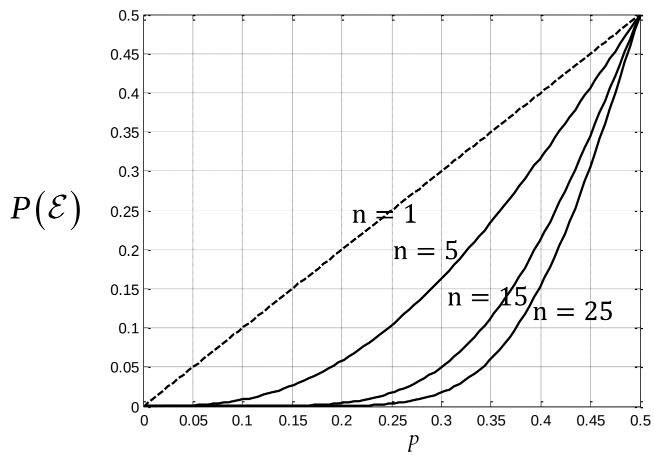


Figure 18: Overall bit error probability for a simple system that uses repetition code at the transmitter (repeat each bit n times) and majority vote at the receiver. The channel is assumed to be binary symmetric with bit error probability p.

Exercise 6.59 (F2011)**.** Kakashi and Gai are eternal rivals. Kakashi is a little stronger than Gai and hence for each time that they fight, the probability that Kakashi wins is 0.55. In a competition, they fight n times (where n is odd). Assume that the results of the fights are independent. The one who wins more will win the competition.

p

Suppose $n = 3$, what is the probability that Kakashi wins the competition.

$$= \binom{3}{3}p^3(1-p)^0 + \binom{3}{2}p^2(1-p)$$

$$p^2 + p(1-p)p + (1-p)pp$$

$$= p^3 + 3p^2(1-p) = 0.575$$

Example 6.60. A stream of bits is transmitted over a binary symmetric channel with crossover probability p.

(a) Consider the first seven bits.

(i) What is the probability that exactly four bits are received in error?

$$\binom{7}{4} p^4 (1-p)^3$$

(ii) What is the probability that at least one bit is received correctly?

$$1 - p^7$$

(b) What is the probability that the *first* error occurs at the fifth bit?

$$(1-p)^4 p$$

(c) What is the probability that the *first* error occurs at the kth bit?

$$(1-p)^{k-1} p \times 1 \times 1 \times 1 \times \ldots$$

(d) What is the probability that the *first* error occurs before or at the kth bit?

$$\sum_{i=1}^{k} (1-p)^{i-1} p = 1 - \sum_{i=k+1}^{\infty} (1-p)^{i-1} p = 1 - \cancel{p} \frac{(1-p)^k}{\underbrace{\cancel{(1-(1-p))}}_{\cancel{p}}}$$

Opposite case: First error occurs after the k^{th} bit.

$\equiv$ the first k bits have no error

$\Rightarrow$ probability $= (1-p)^k$

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \qquad |r| < 1$$

$$\sum_{i=m}^{\infty} r^i = r^m \sum_{i=m}^{\infty} r^{i-m} = r^m \sum_{i=0}^{\infty} r^i = \frac{r^m}{1-r}$$